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Nonlinear superposition formula of the KdV equation with a source

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Abstract. A nonlinear superposition formula of the KdV equation with a source is proved rigorously.

1. Introduction

Recently, the KdV equation with a source, of the form

$$u_t + 6uu_x + u_{xxx} = -\int_{-\infty}^{\infty} \mathrm{d}k\, \nu(|\phi|^2)_x \tag{1a}$$

$$\phi_{xx} + (u + k^2)\phi = 0 \tag{1b}$$

was considered [1–3], where $u = u(x, t)$ and $\phi = \phi(x, t; k)$ are real and complex functions respectively, k is a real parameter and $\nu = \nu(k, t)$ is a given real function. The boundary conditions for u and ϕ are specified as

$$u \to 0 \qquad \text{as } |x| \to \infty$$

$$\phi \to \phi_0\, \mathrm{e}^{\mathrm{i}kx} \qquad \text{as } x \to -\infty$$

where $\phi_0 = \phi_0(k, t)$ is a given function. Equation (1) has already been shown to be integrable by means of the inverse scattering method [1, 2]. By means of the dependent variable transformations $u = 2(\ln f)_{xx}$, $\phi = \phi_0\, \mathrm{e}^{\mathrm{i}kx} g/f$, (1) is transformed into the following bilinear equation [3]

$$D_x(D_t + D_x^3) f \cdot f = -\int_{-\infty}^{\infty} \mathrm{d}k\, \nu |\phi_0|^2 (|g|^2 - f^2) \tag{2a}$$

$$(D_x^2 + 2\mathrm{i}kD_x) g \cdot f = 0 \tag{2b}$$

where $f = f(x, t)$ and $g = g(x, t; k)$ are real and complex functions respectively, and the bilinear operator $D_t^m D_x^n$ is defined by [4, 5]

$$D_t^m D_x^n g \cdot f = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n g(t, x) f(t', x')|_{t'=t, x'=x}.$$

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In [3], Matsuno has presented a two-parameter Bäcklund transformation

$$(D_t + D_x^3 + \mu) f' \cdot f = \frac{i}{4} \int_{-\infty}^{\infty} dk\, \nu |\phi_0|^2 \left(\frac{g'^* g}{k - i\lambda} - \frac{g' g^*}{k + i\lambda} \right) \tag{3a}$$

$$(D_x^2 - 2\lambda D_x) f' \cdot f = 0 \tag{3b}$$

$$D_x g \cdot f' = -(\lambda + ik)(g' f + g f') \tag{3c}$$

$$D_x g' \cdot f = (\lambda - ik)(g' f + g f') \tag{3d}$$

where $\lambda = \lambda(t)$ and $\mu = \mu(t)$ are real Bäcklund parameters and $*$ denotes complex conjugate. A nonlinear superposition formula has also been deduced from the commutability of the BT (3).

The purpose of this paper is to prove the following nonlinear superposition formula of (2) rigorously.

Theorem. Let (f_0, g_0) be a solution of (2) and suppose that (f_i, g_i) $(i = 1, 2)$ is a solution of (2) which is related by (f_0, g_0) under BT (3) with (λ_i, μ_i), i.e. $(f_0, g_0) \xrightarrow{(\lambda_i, \mu_i)} (f_i, g_i)$ $(i = 1, 2)$, $\lambda_1 \neq \pm\lambda_2$, $(\lambda_1 - \lambda_2)_t = 0$, $f_j \neq 0$, $g_j \neq 0 (j = 0, 1, 2)$. Then (f_{12}, g_{12}) defined by

$$\begin{aligned} f_0 f_{12} &= c[D_x - (\lambda_1 - \lambda_2)] f_1 \cdot f_2 \\ g_0 g_{12} &= c[D_x - (\lambda_1 - \lambda_2)] g_1 \cdot g_2 \end{aligned} \qquad c \text{ is a real non-zero constant} \tag{4}$$

is a new solution which is related by (f_1, g_1) and (f_2, g_2) under BT (3) with parameters (λ_2, μ_2), (λ_1, μ_1) respectively.

2. Some lemmas

In order to prove the theorem, we first give some lemmas. In what follows, we always assume that the hypotheses of the theorem are satisfied and (f_{12}, g_{12}) is given by (4).

Lemma 1

$$D_x f_0 \cdot f_{12} = -c(\lambda_1 + \lambda_2) D_x f_1 \cdot f_2 \tag{5}$$

$$c(\lambda_2^2 - \lambda_1^2) f_1 f_2 = [D_x + (\lambda_1 + \lambda_2)] f_0 \cdot f_{12} \tag{6}$$

$$(D_x^2 - 2\lambda_1 D_x) f_{12} \cdot f_2 = 0 \tag{7a}$$

$$(D_x^2 - 2\lambda_2 D_x) f_{12} \cdot f_1 = 0 \tag{7b}$$

$$D_x g_2 \cdot f_{12} = -(\lambda_1 + ik)(g_{12} f_2 + g_2 f_{12}) \tag{8a}$$

$$D_x g_1 \cdot f_{12} = -(\lambda_2 + ik)(g_{12} f_1 + g_1 f_{12}) \tag{8b}$$

$$D_x g_{12} \cdot f_2 = (\lambda_1 - ik)(g_{12} f_2 + g_2 f_{12}) \tag{8c}$$

$$D_x g_{12} \cdot f_1 = (\lambda_2 - ik)(g_{12} f_1 + g_1 f_{12}). \tag{8d}$$

Proof. (5)-(8) can be proved similarly as in [5-7].

Lemma 2

$$D_x^2 f_0 \cdot f_{12} = c[-D_x^3 + 8\lambda_1\lambda_2 D_x + 3(\lambda_1 - \lambda_2) D_x^2] f_1 \cdot f_2. \tag{9}$$

Proof. According to the hypotheses of the theorem and (7*b*), we have

$$
\begin{aligned}
0 &= [(D_x^2-2\lambda_1 D_x)f_1\cdot f_0]f_{12}+[(D_x^2-2\lambda_2 D_x)f_{12}\cdot f_1]f_0 \\
&= 2f_{1xx}f_0f_{12}-2f_{1x}(f_0f_{12})_x+f_1(f_{0xx}f_{12}+f_0f_{12xx}) \\
&\quad +2(\lambda_2-\lambda_1)f_{1x}f_0f_{12}+2f_1(\lambda_1 f_{0x}f_{12}-\lambda_2 f_0 f_{12x}) \\
&\overset{(4)}{=} 2cf_{1xx}[D_x-(\lambda_1-\lambda_2)]f_1\cdot f_2-2cf_{1x}\{[D_x-(\lambda_1-\lambda_2)]f_1\cdot f_2\}_x \\
&\quad +2c(\lambda_2-\lambda_1)f_{1x}[D_x-(\lambda_1-\lambda_2)]f_1\cdot f_2 \\
&\quad +f_1[\tfrac{1}{2}D_x^2 f_0\cdot f_{12}+\tfrac{1}{2}(f_0f_{12})_{xx}+(\lambda_1+\lambda_2)D_xf_0\cdot f_{12}+(\lambda_1-\lambda_2)(f_0f_{12})_x] \\
&= f_1\{-2cf_{1xx}f_{2x}-2c(\lambda_1-\lambda_2)f_{1xx}f_2+2cf_{1x}f_{2xx} \\
&\quad +4c(\lambda_1-\lambda_2)f_{1x}f_{2x}+2c(\lambda_2-\lambda_1)^2f_{1x}f_2 \\
&\quad +\tfrac{1}{2}D_x^2f_0\cdot f_{12}+\tfrac{1}{2}(f_0f_{12})_{xx}+(\lambda_1+\lambda_2)D_xf_0\cdot f_{12}+(\lambda_1-\lambda_2)(f_0f_{12})_x\} \\
&\overset{(4)(15)}{=} f_1\{-2cf_{1xx}f_{2x}-2c(\lambda_1-\lambda_2)f_{1xx}f_2+2cf_{1x}f_{2xx}+4c(\lambda_1-\lambda_2)f_{1x}f_{2x} \\
&\quad +2c(\lambda_2-\lambda_1)^2f_{1x}f_2+\tfrac{1}{2}D_x^2f_0\cdot f_{12}+\tfrac{1}{2}c[(D_x-(\lambda_1-\lambda_2))f_1\cdot f_2]_{xx} \\
&\quad -c(\lambda_1+\lambda_2)^2D_xf_1\cdot f_2+c(\lambda_1-\lambda_2)[(D_x-(\lambda_1-\lambda_2))f_1\cdot f_2]_x\}
\end{aligned}
$$

which implies that (9) holds.

Lemma 3

$$
\begin{aligned}
&-[D_t+\tfrac{1}{4}D_x^3+(\mu_1-\mu_2)]f_1\cdot f_2+\frac{3}{4c(\lambda_1+\lambda_2)}D_x^3f_0\cdot f_{12} \\
&\quad -\frac{i}{4}\int_{-\infty}^{\infty}dk\,\nu|\phi_0|^2\bigg[\frac{D_xg_2\cdot g_1^*}{(k+i\lambda_2)(\lambda_1+ik)}+\frac{D_xg_2^*\cdot g_1}{(k+i\lambda_1)(\lambda_2+ik)} \\
&\quad +\frac{g_1^*g_2}{k-i\lambda_1}+\frac{g_1^*g_2}{k+i\lambda_2}-\frac{g_1g_2^*}{k+i\lambda_1}-\frac{g_1g_2^*}{k-i\lambda_2}\bigg]=0. \qquad (10)
\end{aligned}
$$

Proof. According to the hypotheses of the theorem, we have

$$
\begin{aligned}
0 &= \bigg[(D_t+D_x^3-\mu_1)f_0\cdot f_1+\frac{i}{4}\int_{-\infty}^{\infty}dk\,\nu|\phi_0|^2\bigg(\frac{g_1^*g_0}{k-i\lambda_1}-\frac{g_1g_0^*}{k+i\lambda_1}\bigg)\bigg]f_2 \\
&\quad -\bigg[(D_t+D_x^3-\mu_2)f_0\cdot f_2+\frac{i}{4}\int_{-\infty}^{\infty}dk\,\nu|\phi_0|^2\bigg(\frac{g_2^*g_0}{k-i\lambda_2}-\frac{g_2g_0^*}{k+i\lambda_2}\bigg)\bigg]f_1 \\
&\overset{(A1,A2)}{=} -f_0D_tf_1\cdot f_2-3f_{0xx}D_xf_1\cdot f_2+3f_{0x}(D_xf_1\cdot f_2)_x \\
&\quad -\tfrac{1}{4}f_0[D_x^3f_1\cdot f_2+3(D_xf_1\cdot f_2)_{xx}]+(\mu_2-\mu_1)f_0f_1f_2 \\
&\quad +\frac{i}{4}\int_{-\infty}^{\infty}dk\,\nu|\phi_0|^2\bigg\{\frac{g_1^*}{k-i\lambda_1}\cdot\frac{1}{\lambda_2-ik}[D_x-(\lambda_2-ik)]g_2\cdot f_0 \\
&\quad -\frac{g_1}{k+i\lambda_1}\cdot\frac{1}{\lambda_2+ik}[D_x-(\lambda_2+ik)]g_2^*\cdot f_0
\end{aligned}
$$

$$-\frac{g_2^*}{k-\mathrm{i}\lambda_2}\cdot\frac{1}{\lambda_1-\mathrm{i}k}[D_x-(\lambda_1-\mathrm{i}k)]g_1\cdot f_0$$

$$\left.+\frac{g_2}{k+\mathrm{i}\lambda_2}\cdot\frac{1}{\lambda_1+\mathrm{i}k}[D_x-(\lambda_1+\mathrm{i}k)]g_1^*\cdot f_0\right\}$$

$$\overset{(5)}{=} f_0\{-D_tf_1\cdot f_2-\tfrac{1}{4}[D_x^3f_1\cdot f_2+3(D_xf_1\cdot f_2)_{xx}]+(\mu_2-\mu_1)f_1f_2\}$$

$$+\frac{\mathrm{i}}{4}\int_{-\infty}^{\infty}\mathrm{d}k\,\nu|\phi_0|^2\left\{\frac{1}{(k+\mathrm{i}\lambda_2)(\lambda_1+\mathrm{i}k)}[(D_xf_0\cdot g_2)g_1^*-(D_xf_0\cdot g_1^*)g_2]\right.$$

$$+\frac{1}{(k+\mathrm{i}\lambda_1)(\lambda_2+\mathrm{i}k)}[(D_xf_0\cdot g_2^*)g_1-(D_xf_0\cdot g_1)g_2^*]$$

$$\left.+f_0\left[-\frac{g_1^*g_2}{k-\mathrm{i}\lambda_1}+\frac{g_1g_2^*}{k+\mathrm{i}\lambda_1}+\frac{g_1g_2^*}{k-\mathrm{i}\lambda_2}-\frac{g_1^*g_2}{k+\mathrm{i}\lambda_2}\right]\right\}$$

$$+\frac{3}{c(\lambda_1+\lambda_2)}f_{0xx}D_xf_0\cdot f_{12}-\frac{3}{c(\lambda_1+\lambda_2)}f_{0x}(D_xf_0\cdot f_{12})_x$$

$$\overset{(A1)(15)}{=} f_0\left\{[-D_t-\tfrac{1}{4}D_x^3+(\mu_2-\mu_1)]f_1\cdot f_2\right.$$

$$+\frac{\mathrm{i}}{4}\int_{-\infty}^{\infty}\mathrm{d}k\,\nu|\phi_0|^2\left[-\frac{D_xg_2\cdot g_1^*}{(k+\mathrm{i}\lambda_2)(\lambda_1+\mathrm{i}k)}-\frac{D_xg_2^*\cdot g_1}{(k+\mathrm{i}\lambda_1)(\lambda_2+\mathrm{i}k)}\right.$$

$$\left.-\frac{g_1^*g_2}{k-\mathrm{i}\lambda_1}+\frac{g_1g_2^*}{k+\mathrm{i}\lambda_1}+\frac{g_1g_2^*}{k-\mathrm{i}\lambda_2}-\frac{g_1^*g_2}{k+\mathrm{i}\lambda_2}\right]$$

$$\left.-\frac{3}{4c(\lambda_1+\lambda_2)}(D_xf_0\cdot f_{12})_{xx}-\frac{3}{c(\lambda_1+\lambda_2)}f_{0xx}f_{12x}+\frac{3}{c(\lambda_1+\lambda_2)}f_{0x}f_{12xx}\right\}$$

which implies that (10) holds.

Lemma 4

$$\frac{1}{4c(\lambda_1+\lambda_2)}D_x^3f_0\cdot f_{12}=[\tfrac{3}{4}D_x^3-4\lambda_1\lambda_2D_x-2(\lambda_1-\lambda_2)D_x^2]f_1\cdot f_2. \tag{11}$$

Proof. According to the hypotheses of the theorem, we have

$$0=[(D_x^2-2\lambda_1D_x)f_1\cdot f_0]_xf_2-[(D_x^2-2\lambda_2D_x)f_2\cdot f_0]_xf_1$$

$$\overset{(A3)}{=}-f_{0xx}D_xf_1\cdot f_2-f_{0x}(D_xf_1\cdot f_2)_x+\tfrac{1}{4}f_0[D_x^3f_1\cdot f_2+3(D_xf_1\cdot f_2)_{xx}]$$

$$+2(\lambda_1-\lambda_2)f_{0xx}f_1f_2-2f_0(\lambda_1f_{1xx}f_2-\lambda_2f_1f_{2xx})$$

$$\overset{(5,6)}{=}\frac{1}{c(\lambda_1+\lambda_2)}f_{0xx}D_xf_0\cdot f_{12}+\frac{1}{c(\lambda_1+\lambda_2)}f_{0x}(D_xf_0\cdot f_{12})_x$$

$$+\tfrac{1}{4}f_0[D_x^3f_1\cdot f_2+3(D_xf_1\cdot f_2)_{xx}]-\frac{2}{c(\lambda_1+\lambda_2)}f_{0xx}[D_x+(\lambda_1+\lambda_2)]f_0\cdot f_{12}$$

$$-f_0\{(\lambda_1+\lambda_2)(D_xf_1\cdot f_2)_x+\tfrac{1}{2}(\lambda_1-\lambda_2)[D_x^2f_1\cdot f_2+(f_1f_2)_{xx}]\}$$

$$\overset{(5,6)}{=} f_0\left\{\frac{1}{c(\lambda_1+\lambda_2)} f_{0xx}f_{12x} - \frac{1}{c(\lambda_1+\lambda_2)} f_{0x}f_{12xx} + \tfrac{1}{4}D_x^3 f_1\cdot f_2 \right.$$
$$- \frac{3}{4c(\lambda_1+\lambda_2)}(D_x f_0\cdot f_{12})_{xx} - \frac{2}{c} f_{0xx}f_{12} + \frac{1}{c}(D_x f_0\cdot f_{12})_x$$
$$\left. - \tfrac{1}{2}(\lambda_1-\lambda_2)D_x^2 f_1\cdot f_2 + \frac{1}{2c(\lambda_1+\lambda_2)}[(D_x+(\lambda_1+\lambda_2))f_0\cdot f_{12}]_{xx}\right\}$$
$$= f_0\left\{-\frac{1}{4c(\lambda_1+\lambda_2)} D_x^3 f_0\cdot f_{12} + \tfrac{1}{4}D_x^3 f_1\cdot f_2 - \frac{1}{2c} D_x^2 f_0\cdot f_{12} - \tfrac{1}{2}(\lambda_1-\lambda_2)D_x^2 f_1\cdot f_2\right\}$$

which implies that (11) holds by using (9).

Lemma 5

$$[-D_t + 2D_x^3 - 12\lambda_1\lambda_2 D_x - 6(\lambda_1-\lambda_2)D_x^2 + (\mu_2-\mu_1)]f_1\cdot f_2$$
$$- \frac{\mathrm{i}}{4}\int_{-\infty}^{\infty} \mathrm{d}k\, \nu|\phi_0|^2\left[\frac{D_x g_2\cdot g_1^*}{(k+\mathrm{i}\lambda_2)(\lambda_1+\mathrm{i}k)} + \frac{D_x g_2^*\cdot g_1}{(k+\mathrm{i}\lambda_1)(\lambda_2+\mathrm{i}k)}\right.$$
$$\left. + \frac{g_1^* g_2}{k-\mathrm{i}\lambda_1} + \frac{g_1^* g_2}{k+\mathrm{i}\lambda_2} - \frac{g_1 g_2^*}{k+\mathrm{i}\lambda_1} - \frac{g_1 g_2^*}{k-\mathrm{i}\lambda_2}\right] = 0. \tag{12}$$

Proof. Equation (12) can be deduced directly from (10) and (11).

Lemma 6

$$D_x^3(D_x f_1\cdot f_2)\cdot f_1 f_2 = -D_x\{[2D_x^3 - 12\lambda_1\lambda_2 D_x - 6(\lambda_1-\lambda_2)D_x^2]f_1\cdot f_2\}\cdot f_1 f_2 \tag{13}$$
$$D_x(D_x^2 f_1\cdot f_2)\cdot(D_x f_1\cdot f_2) = D_x\{[D_x^3 - 4\lambda_1\lambda_2 D_x - 2(\lambda_1-\lambda_2)D_x]f_1\cdot f_2\}\cdot f_1 f_2 \tag{14}$$

Proof

$$c^2(\lambda_1^2-\lambda_2^2)D_x^3(D_x f_1\cdot f_2)\cdot f_1 f_2$$
$$\overset{(5,6)}{=} \frac{1}{(\lambda_1+\lambda_2)} D_x^3(D_x f_0\cdot f_{12})\cdot[(D_x+(\lambda_1+\lambda_2))f_0\cdot f_{12}]$$
$$\overset{(A4)}{=} D_x^3(D_x f_0\cdot f_{12})\cdot f_0 f_{12}$$
$$\overset{(A5)}{=} D_x(D_x^3 f_0\cdot f_{12})\cdot f_0 f_{12} + 3D_x(D_x f_0\cdot f_{12})\cdot(D_x^2 f_0\cdot f_{12})$$
$$\overset{(5,9,11)}{=} 4c^2(\lambda_1+\lambda_2)D_x\{[\tfrac{3}{4}D_x^3 - 4\lambda_1\lambda_2 D_x$$
$$- 2(\lambda_1-\lambda_2)D_x^2]f_1\cdot f_2\}\cdot\{[D_x-(\lambda_1-\lambda_2)]f_1\cdot f_2\}$$
$$- 3c^2(\lambda_1+\lambda_2)D_x(D_x f_1\cdot f_2)\cdot\{[-D_x^3 + 8\lambda_1\lambda_2 D_x + 3(\lambda_1-\lambda_2)D_x^2]f_1\cdot f_2\}$$
$$= -4c^2(\lambda_1^2-\lambda_2^2)D_x\{[\tfrac{3}{4}D_x^3 - 4\lambda_1\lambda_2 D_x - 2(\lambda_1-\lambda_2)D_x^2]f_1\cdot f_2\}\cdot f_1 f_2$$
$$+ c^2(\lambda_1^2-\lambda_2^2)D_x(D_x^2 f_1\cdot f_2)\cdot(D_x f_1\cdot f_2)$$

which implies that (13) and (14) hold by using (A5).

Lemma 7

$$D_x g_1 g_2^*\cdot(D_x f_1\cdot f_2) = D_x(D_x g_1\cdot g_2^*)\cdot f_1 f_2 + 2\mathrm{i}k D_x g_1 g_2^*\cdot f_1 f_2 \tag{15}$$
$$D_x(D_x f_1\cdot f_2)\cdot g_2 g_1^* = -D_x(D_x g_1^*\cdot g_2)\cdot f_1 f_2 + 2\mathrm{i}k D_x g_2 g_1^*\cdot f_1 f_2. \tag{16}$$

Proof. By the use of (A6), we have

$$\begin{aligned} D_x g_1 g_2^* \cdot (D_x f_1 \cdot f_2) - D_x(D_x g_1 \cdot g_2^*) \cdot f_1 f_2 \\ &= D_x[(D_x g_2^* \cdot g_1) \cdot f_1 f_2 + g_2^* g_1 \cdot (D_x f_1 \cdot f_2)] \\ &= (D_x^2 g_2^* \cdot f_2) f_1 g_1 - (D_x^2 f_1 \cdot g_1) f_2 g_2^*. \end{aligned} \tag{17}$$

On the other hand, since (f_1, g_1) and (f_2, g_2) are two solutions of (2), we have

$$(D_x^2 - 2ikD_x) g_2^* \cdot f_2 = 0 \tag{18}$$

$$(D_x^2 + 2ikD_x) g_1 \cdot f_1 = 0. \tag{19}$$

By the use of (18) and (19), (17) becomes

$$\begin{aligned} D_x g_1 g_2^* \cdot (D_x f_1 \cdot f_2) - D_x(D_x g_1 \cdot g_2^*) \cdot f_1 f_2 \\ &= 2ik(D_x g_2^* \cdot f_2) f_1 g_1 + 2ik(D_x g_1 \cdot f_1) f_2 g_2^* \\ &\overset{(A7)}{=} 2ik D_x g_2^* g_1 \cdot f_1 f_2 \end{aligned}$$

which implies that (15) holds. Further, (16) can be deduced directly from (15).

3. Proof of the theorem

Based on lemma 1, it suffices to show that

$$Q_1 \equiv (D_t + D_x^3 + \mu_2) f_{12} \cdot f_1 - \frac{i}{4} \int_{-\infty}^{\infty} dk\, \nu |\phi_0|^2 \left(\frac{g_{12}^* g_1}{k - i\lambda_2} - \frac{g_{12} g_1^*}{k + i\lambda_2} \right) = 0$$

$$Q_2 \equiv (D_t + D_x^3 + \mu_1) f_{12} \cdot f_2 - \frac{i}{4} \int_{-\infty}^{\infty} dk\, \nu |\phi_0|^2 \left(\frac{g_{12}^* g_2}{k - i\lambda_1} - \frac{g_{12} g_2^*}{k + i\lambda_1} \right) = 0.$$

By the use of lemmas 2-7, we have

$$\begin{aligned} Q_1 f_0 f_2 &= \left[(D_t + D_x^3 - \mu_2) f_0 \cdot f_2 + \frac{i}{4} \int_{-\infty}^{\infty} dk\, \nu |\phi_0|^2 \left(\frac{g_2^* g_0}{k - i\lambda_2} - \frac{g_2 g_0^*}{k + i\lambda_2} \right) \right] f_1 f_{12} \\ &\quad - \left[(D_t + D_x^3 - \mu_2) f_1 \cdot f_{12} + \frac{i}{4} \int_{-\infty}^{\infty} dk\, \nu |\phi_0|^2 \left(\frac{g_{12}^* g_1}{k - i\lambda_2} - \frac{g_{12} g_1^*}{k + i\lambda_2} \right) \right] f_0 f_2 \\ &\overset{(A7, A8)}{=} D_t f_0 f_{12} \cdot f_1 f_2 + \tfrac{1}{4} D_x^3 f_0 f_{12} \cdot f_1 f_2 \\ &\quad + \tfrac{3}{4} D_x[(D_x^2 f_0 \cdot f_{12}) \cdot f_1 f_2 + f_0 f_{12} \cdot (D_x^2 f_1 \cdot f_2) + 2(D_x f_0 \cdot f_{12}) \cdot (D_x f_1 \cdot f_2)] \\ &\quad + \frac{i}{4} \int_{-\infty}^{\infty} dk\, \nu |\phi_0|^2 \left\{ \frac{g_2^* f_{12}}{k - i\lambda_2} \frac{[D_x - (\lambda_1 - ik)] g_1 \cdot f_0}{\lambda_1 - ik} \right. \\ &\quad - \frac{g_2 f_{12}}{k + i\lambda_2} \frac{[D_x - (\lambda_1 + ik)] g_1^* \cdot f_0}{\lambda_1 + ik} \\ &\quad \left. + \frac{g_1 f_0}{k - i\lambda_2} \frac{[D_x + (\lambda_1 - ik)] g_2^* \cdot f_{12}}{\lambda_1 - ik} - \frac{g_1^* f_0}{k + i\lambda_2} \frac{[D_x + (\lambda_1 + ik)] g_2 \cdot f_{12}}{\lambda_1 + ik} \right\} \\ &\overset{(4, 5, 9, A4)}{=} c D_t (D_x f_1 \cdot f_2) \cdot f_1 f_2 + \frac{c}{4} D_x^3 (D_x f_1 \cdot f_2) \cdot f_1 f_2 \end{aligned}$$

$$+\frac{3c}{4}D_x\{[-D_x^3+8\lambda_1\lambda_2D_x+3(\lambda_1-\lambda_2)D_x^2]f_1\cdot f_2\}\cdot f_1f_2$$

$$+\frac{3c}{4}D_x\{[D_x-(\lambda_1-\lambda_2)]f_1\cdot f_2\}\cdot(D_x^2f_1\cdot f_2)$$

$$+\frac{i}{4}\int_{-\infty}^{\infty}dk\,\nu|\phi_0|^2\left\{\frac{D_xg_1g_2^*\cdot f_0f_{12}}{(k-i\lambda_2)(\lambda_1-ik)}+\frac{D_xf_0f_{12}\cdot g_2g_1^*}{(k+i\lambda_2)(\lambda_1+ik)}\right\}$$

$$\overset{(13,14,4)}{=}cD_t(D_xf_1\cdot f_2)\cdot f_1f_2-\frac{c}{4}D_x\{[2D_x^3-12\lambda_1\lambda_2D_x-6(\lambda_1-\lambda_2)D_x^2]f_1\cdot f_2\}\cdot f_1f_2$$

$$+\frac{3c}{4}D_x\{[-D_x^3+8\lambda_1\lambda_2D_x+3(\lambda_1-\lambda_2)D_x^2]f_1\cdot f_2\}\cdot f_1f_2$$

$$-\frac{3c}{4}D_x\{[D_x^3-4\lambda_1\lambda_2D_x-2(\lambda_1-\lambda_2)D_x^2]f_1\cdot f_2\}\cdot f_1f_2$$

$$+\frac{i}{4}\int_{-\infty}^{\infty}dk\,\nu|\phi_0|^2\left\{\frac{D_xg_1g_2^*\cdot c[(D_x-(\lambda_1-\lambda_2))f_1\cdot f_2]}{(k-i\lambda_2)(\lambda_1-ik)}\right.$$

$$\left.+\frac{cD_x[(D_x-(\lambda_1-\lambda_2))f_1\cdot f_2]\cdot g_2g_1^*}{(k+i\lambda_2)(\lambda_1+ik)}\right\}$$

$$\overset{(15,16)}{=}cD_t(D_xf_1\cdot f_2)\cdot f_1f_2-2cD_x\{[D_x^3-6\lambda_1\lambda_2D_x-3(\lambda_1-\lambda_2)D_x^2]f_1\cdot f_2\}\cdot f_1f_2$$

$$+\frac{i}{4}c\int_{-\infty}^{\infty}dk\,\nu|\phi_0|^2\left\{\frac{D_x(D_xg_1\cdot g_2^*)\cdot f_1f_2+2ikD_xg_1g_2^*\cdot f_1f_2-(\lambda_1-\lambda_2)D_xg_1g_2^*\cdot f_1f_2}{(k-i\lambda_2)(\lambda_1-ik)}\right.$$

$$\left.+\frac{-D_x(D_xg_1^*\cdot g_2)\cdot f_1f_2+2ikD_xg_2g_1^*\cdot f_1f_2+(\lambda_1-\lambda_2)D_xg_1^*g_2\cdot f_1f_2}{(k+i\lambda_2)(\lambda_1+ik)}\right\}$$

$$=cD_x\left\{[D_t-2D_x^3+12\lambda_1\lambda_2D_x+6(\lambda_1-\lambda_2)D_x^2]f_1\cdot f_2\right.$$

$$+\frac{i}{4}\int_{-\infty}^{\infty}dk\,\nu|\phi_0|^2\left[\frac{D_xg_2\cdot g_1^*}{(k+i\lambda_2)(\lambda_1+ik)}+\frac{D_xg_2^*\cdot g_1}{(k+i\lambda_1)(\lambda_2+ik)}\right.$$

$$\left.\left.+\frac{g_1^*g_2}{k-i\lambda_1}+\frac{g_1^*g_2}{k+i\lambda_2}-\frac{g_1g_2^*}{k+i\lambda_1}-\frac{g_1g_2^*}{k-i\lambda_2}\right]\right\}\cdot f_1f_2$$

$$\overset{(12,A4)}{=}0$$

which implies that $Q_1=0$. Similarly, we can prove $Q_2=0$. Thus we have completed the proof of the theorem. □

Remark. We can check that (f_{12}, g_{12}) given by (4) is still a solution of (2) as $\lambda_1^2=\lambda_2^2$ and $c\neq 0$ if the integration constants are suitably chosen. In fact, we have not imposed any restrictions on the parameter λ for proving (6). So we get

$$[D_x+(\lambda_1+\lambda_2)]f_0\cdot f_{12}=0$$

i.e.

$$\left(\ln \frac{f_{12}}{f_0}\right)_x = (\lambda_1 + \lambda_2).$$

Thus

$$f_{12} = f_0\, e^{(\lambda_1+\lambda_2)x + g_1(t)}$$

where $g_1(t)$ is an arbitrary function of t. Similarly, we have

$$g_{12} = g_0\, e^{(\lambda_1+\lambda_2)x + g_2(t)}$$

where $g_2(t)$ is an arbitrary function of t. Now we choose $g_1(t) = g_2(t)$. In this case, by a direct computation, we can check that (f_{12}, g_{12}) indeed satisfies (2) and the corresponding $(u^{12}, \phi^{12}) \equiv (u^0, \phi^0)$. So this situation is a trivial one.

4. Concluding remark

We have proved the nonlinear superposition formula of the KdV equation with a source (2). It is noticed that in [8] we consider the following bilinear KdV hierarchy with a unified form

$$(D_x D_{t_{2n+1}} + \tfrac{2}{3} D_{t_{2n-1}} D_x^3 - \tfrac{1}{3} D_{t_3} D_{t_{2n-1}}) f \cdot f = 0 \qquad n \in \mathbb{Z}_+,\ t_1 = x. \tag{20}$$

A nonlinear superposition formula of (20) is also proved rigorously. Similarly, we can consider the following KdV hierarchy with a source

$$(D_x D_{t_{2n+1}} + \tfrac{2}{3} D_{t_{2n-1}} D_x^3 - \tfrac{1}{3} D_{t_3} D_{t_{2n-1}}) f \cdot f = -\int_{-\infty}^{\infty} \mathrm{d}k\, \nu |\phi_0|^2 (|g|^2 - f^2) \qquad n \in \mathbb{Z}_+,\ t_1 = x$$

$$(D_x^2 + 2ikD_x) g \cdot f = 0.$$

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Appendix

The following bilinear operator identities hold for arbitrary functions a, b, c and d:

$$(D_t a \cdot b) c - (D_t a \cdot c) b = -a D_t b \cdot c \tag{A1}$$

$$(D_x^3 a \cdot b) c - (D_x^3 a \cdot c) b = -3 a_{xx} D_x b \cdot c + 3 a_x (D_x b \cdot c)_x - \tfrac{1}{4} a [D_x^3 b \cdot c + 3 (D_x b \cdot c)_{xx}] \tag{A2}$$

$$(D_x^2 a \cdot b)_x c - (D_x^2 a \cdot c)_x b = -a_{xx} D_x b \cdot c - a_x (D_x b \cdot c)_x + \tfrac{1}{4} a [D_x^3 b \cdot c + 3 (D_x b \cdot c)_{xx}] \tag{A3}$$

$$D_x^{2n+1} a \cdot a = 0 \qquad n \in \mathbb{Z}_+ \cup \{0\} \tag{A4}$$

$$D_x^3 (D_x a \cdot b) \cdot ab = D_x (D_x^3 a \cdot b) \cdot ab + 3 D_x (D_x a \cdot b) \cdot (D_x^2 a \cdot b) \tag{A5}$$

$$D_x [(D_x a \cdot d) \cdot cb + ad \cdot (D_x c \cdot b)] = (D_x^2 a \cdot b) cd - ab D_x^2 c \cdot d \tag{A6}$$

$$(D_x a\cdot b)cd - abD_x c\cdot d = D_x ad\cdot cb \tag{A7}$$

$$(D_x^3 a\cdot b)cd - abD_x^3 c\cdot d = \tfrac{1}{4}D_x^3 ad\cdot cb + \tfrac{3}{4}D_x[(D_x^2 a\cdot d)\cdot cb + 2(D_x a\cdot d)\cdot(D_x c\cdot b) + ad\cdot(D_x^2 c\cdot b)]. \tag{A8}$$

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